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FURTHER CONSIDERATIONS ON THE PROBLEM OF TORSION AND FLEXURE OF PRISMATICAL BEAMS

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Abstract—We establish a linear system of one dimensional beam equations, through use of the principle of minimum potential energy in conjunction with suitable displacement approximations, which takes account of warping stiffness in addition to stretching, transverse shearing, bending and twisting stiffness. We use this system of equations for the asymptotic analysis of a warping boundary layer, including the concept of contracted and reduced boundary conditions for the determination of the state of the beam outside the layer, as well as for the derivation of expressions for cross sectional shear center coordinates which are more general than previous expressions for these quantities.

INTRODUCTION

In what follows we supplement two recent considerations of the theory of unsymmetrical cross section beams, with particular emphasis on the extent to which the concept of warping stiffness, in addition to the concepts of stretching, shearing, bending and twisting stiffness, is an essential part of this theory[1, 3].

In [1] one of our main concerns was the problem of determining shear and twist center locations through use of the principle of minimum potential energy in conjunction with suitable approximations for components of displacement. In [3] one of our main concerns was the asymptotic solution of the problem of cantilever torsion and flexure for a restricted class of beams, allowing the use of the theory of shear deformable plates, through the utilization of the concept of edge zone and interior solution portions of a differential bending moment function which is part of the theory of this class of unsymmetrical cross section beams.

The present consideration of the potential energy procedure for the general cross section beam is at the same time simpler and less restrictive than the work in [1]. The restrictive assumption which is now not made is the assumption of negligible translational deflections due to transverse shear deformability. Aside from the simplification which results from this we deduce, in particular, conclusions not previously stated regarding the nature of the problem of appropriately choosing a description of the state of cross sectional warping.

Given the general seventh order system of one-dimensional beam equations accounting for the effects of stretching, shearing, bending, twisting and warping, we proceed, in extension of the work in [3], to the derivation of a one-dimensional compatibility equation, so as to allow a sequential determination of statical and kinematical quantities in the seventh order theory, in analogy to what is automatically possible in the classical sixth order theory without account for the effect of warping stiffness. It is thought that the idea of this compatibility equation and the use of it for the recognition of distinct edge zone and interior solution contributions, with the ensuing possibility of a derivation of contracted kinematical boundary conditions for an asymptotic sequential determination of interior and edge zone states, are what is most interesting in the present analysis.

DERIVATION OF ONE-DIMENSIONAL BEAM EQUATIONS

We begin as in [1] by stipulating as approximations for components of displacement

$$\bar{u} = u(z) - y\,\theta(z), \quad \bar{v} = v(z) + x\,\theta(z), \tag{1a, b}$$

$$\bar{w} = w(z) + x\,\alpha(z) + y\,\beta(z) + g(x, y)\lambda(z). \tag{2}$$

In this x and y are coordinates in the cross section, the direction of z coincides with the axis of the beam, and g is to be *assumed* suitably.

We determine the seven functions of z in (1) and (2) through use of the principle of minimum potential energy, with the energy functional here taken in the form

$$I = \frac{1}{2}\int (E\tilde{\epsilon}_z^2 + G\tilde{\gamma}_{xz}^2 + G\tilde{\gamma}_{yz}^2)\,\mathrm{d}V - \int (fw + pu + qv + m\alpha + n\beta + t\theta + r\lambda)\,\mathrm{d}z. \tag{3}$$

In this f, p, q, m, n, t and r are one-dimensional components of load intensity due to body forces and/or surface tractions and

$$\tilde{\epsilon}_z = \tilde{w}_z = w' + x\alpha' + y\beta' + g\lambda', \tag{4}$$

$$\tilde{\gamma}_{xz} = \tilde{u}_{,z} + \tilde{w}_{,x} = u' + \alpha - y\theta' + g_{,x}\lambda, \tag{5}$$

$$\tilde{\gamma}_{yz} = \tilde{v}_{,z} + \tilde{w}_{,y} = v' + \beta + x\theta' + g_{,y}\lambda. \tag{6}$$

We introduce (4)–(6) into the variational equation $\delta I = 0$ and write this equation in the form

$$\int [F\delta w' + M\delta\alpha' + N\delta\beta' + R\delta\lambda' + P\delta(u' + \alpha) + Q\delta(v' + \beta) + T\delta\theta' + S\delta\lambda]\,\mathrm{d}z$$

$$- \int (f\delta w + m\delta\alpha + n\delta\beta + r\delta\lambda + p\delta u + q\delta v + t\delta\theta)\,\mathrm{d}z = 0, \tag{7}$$

with F, P, Q being cross sectional forces, M, N, T cross sectional moments and R, S supplementary cross sectional stress measures.

Equation (7) implies the seven one-dimensional Euler equilibrium differential equations

$$F' + f = 0, \quad M' - p + m = 0, \quad N' - Q + n = 0, \tag{8a–c}$$

$$P' + p = 0, \quad Q' + q = 0, \quad T' + t = 0, \quad R' - S + r = 0, \tag{9a–d}$$

with eqns (4)–(6) implying the system of one-dimensional constitutive equations

$$\begin{bmatrix} F \\ M \\ N \\ R \end{bmatrix} = \begin{bmatrix} D_{ww} & D_{w\alpha} & D_{w\beta} & D_{w\lambda} \\ D_{\alpha w} & \cdot & \cdot & D_{\alpha\lambda} \\ \cdot & \cdot & \cdot & \\ D_{\lambda w} & \cdot & \cdot & D_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} w' \\ \alpha' \\ \beta' \\ \lambda' \end{bmatrix}, \quad \begin{bmatrix} P \\ Q \\ T \\ S \end{bmatrix} = \begin{bmatrix} C_{uu} & C_{uv} & C_{u\theta} & C_{u\lambda} \\ C_{vu} & \cdot & \cdot & C_{v\lambda} \\ C_{\theta u} & \cdot & C_{\theta\theta} & C_{\theta\lambda} \\ C_{\lambda u} & \cdot & \cdot & C_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} u' + \alpha \\ v' + \beta \\ \theta' \\ \lambda \end{bmatrix} \tag{10}$$

where the D and C in (10) are of the form

$$D_{ww} = \int E d_{ww}\,\mathrm{d}A, \quad C_{uu} = \int G c_{uu}\,\mathrm{d}A, \text{etc.} \tag{11}$$

and where the d and c in (11) are as shown below

d	w	α	β	λ
w	1	x	y	g
α	x	x^2	xy	xg
β	y	yx	y^2	yg
λ	g	gx	gy	g^2

c	u	v	θ	λ
u	1	0	-y	$g_{,x}$
v	0	1	x	$g_{,y}$
θ	-y	x	x^2+y^2	$xg_{,y} - yg_{,x}$
λ	$g_{,x}$	$g_{,y}$	$xg_{,y}-yg_{,x}$	$g_{,x}^2+g_{,y}^2$

(12)

CHOICE OF THE FUNCTION g

Given that the system of beam equations (8)–(10) includes as (trivial) special cases the appropriate equations for pure stretching and bending of prismatical beams under the influence of end forces and end moments, as exact consequences of the three-dimensional theory of elasticity, it suggests itself to choose the function g in such a way that the same is true insofar as the St. Venant theory of torsion due to applied end torques is concerned. For this to be the case the function g must be the warping function ϕ of St. Venant torsion. Assuming, in extension of the classical theory, a shear modulus $G = G(x, y)$ we have then as conditions determining the function $g = \phi$, the differential equation

$$[G(\phi_{,x} - y)]_{,x} + [G(\phi_{,y} + x)]_{,y} = 0, \tag{13}$$

in conjunction with the boundary condition of vanishing wall tractions

$$(\phi_{,x} - y)\,\mathrm{d}y - (\phi_{,y} + x)\,\mathrm{d}x = 0. \tag{14}$$

Equations (13) and (14) imply, in generalization of well known results for the case $G = \text{const}$, the validity of the three conditions

$$\int G(\phi_{,x} - y)\,\mathrm{d}A = 0, \quad \int G(\phi_{,y} + x)\,\mathrm{d}A = 0, \tag{15a, b}$$

and

$$\int G(y\phi_{,x} - x\phi_{,y})\,\mathrm{d}A = \int G(\phi_{,x}^2 + \phi_{,y}^2)\,\mathrm{d}A. \tag{16}$$

It is convenient for what follows to determine the arbitrary constant which remains in ϕ by stipulating that

$$\int E\phi\,\mathrm{d}A = 0, \tag{17}$$

and to introduce the defining relations

$$(A_g, S_{xG}, S_{yG}, I_G, J_G) = \int (1, x, y, x^2 + y^2, \phi_{,x}^2 + \phi_{,y}^2)\, G\,\mathrm{d}A, \tag{18}$$

$$(A_E, S_{xE}, S_{yE}, I_{xE}, I_{yE}, K_E) = \int (1, x, y, x^2, y^2, xy) E\,\mathrm{d}A, \tag{19}$$

$$(\Gamma_{xE}, \Gamma_{yE}, \Gamma_{\phi E}) = \int (x, y, \phi)\,\phi E\,\mathrm{d}A \tag{20}$$

The matrices $\mathbf{D}$ and $\mathbf{C}$ in (10) and (11) therewith assume the form†

$$\mathbf{D} = \begin{bmatrix} A_E & S_{xE} & S_{yE} & 0 \\ S_{xE} & I_{xE} & K_E & \Gamma_{xE} \\ S_{yE} & K_E & I_{yE} & \Gamma_{yE} \\ 0 & \Gamma_{xE} & \Gamma_{yE} & \Gamma_{\phi E} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} A_G & 0 & -S_{yG} & S_{yG} \\ 0 & A_G & S_{xG} & -S_{xG} \\ -S_{yG} & S_{xG} & I_G & -J_G \\ S_{yG} & -S_{xG} & -J_G & J_G \end{bmatrix}. \tag{21}$$

†We note in this connection that it should read $\Gamma_\phi = \int \phi^2 E\,\mathrm{d}S$ in place of $\Gamma = \int \phi E\,\mathrm{d}S$ in equation (26b) in [1], with Γ_ϕ in place of Γ in eqns (27), (29)–(31) and (34).

AN ALTERNATE CHOICE OF g

Considering the relative complexity of the problem of determining $g = \phi$ in accordance with (13) and (14) the following alternate choice of g is "reasonable". We take account of the fact that (13) and (14) are Euler differential equations and Euler boundary conditions of the variational equation

$$\delta \int [G(g_{,x} - y)^2 + G(g_{,y} + x)^2]\,\mathrm{d}A = 0. \tag{22}$$

We use (22) in order to determine an approximation to ϕ, in the form

$$g = c_0 + c_1 x + c_2 y + \frac{1}{2} c_3 x^2 + \frac{1}{2} c_4 y^2 + c_5 xy. \tag{23}$$

We find by introducing (23) into (22) as a system of five equations for the determination of the coefficients c_1 to c_5,

$$c_1 A_G + c_3 S_{xG} + c_5 S_{yG} = S_{yG}, \tag{24a}$$

$$c_1 S_{xG} + c_3 I_{xG} + c_5 K_G = K_G, \tag{24b}$$

$$c_2 A_G + c_4 S_{yG} + c_5 S_{xG} = -S_{xG}, \tag{24c}$$

$$c_2 S_{xG} + c_4 I_{yG} + c_5 K_G = -K_G, \tag{24d}$$

$$c_1 S_{yG} + c_2 S_{xG} + (c_3 + c_4) K_G + c_5 I_G = I_{yG} - I_{xG}. \tag{24e}$$

Having then the values of c_1 to c_5 in terms of cross sectional weighted averages of G we may determine the elements of the two coefficient matrices in (10) and (11) in terms of these weighted averages in accordance with the defining relations in (11) and (12), in conjunction with the additional stipulation

$$\int Eg\,\mathrm{d}A = 0, \tag{17'}$$

for the purpose of determining the sixth coefficient c_0.

As regards this determination for the general unsymmetric cross section case we only note here the fact that we retain in the approximate analysis the coefficient property $C_{\lambda\lambda} = -C_{\theta\lambda}$ which was encountered in the result for $g = \phi$ as stated in (21). To see that this is so it is only necessary to set $\delta g_{,x} = g_{,x}$ and $\delta g_{,y} = g_{,y}$ in the relation

$$\int [G(g_{,x} - y)\delta g_{,x} + G(g_{,y} + x)\delta g_{,y}]\,\mathrm{d}A = 0, \tag{22'}$$

which occurs in the evaluation of (22). Having this result we have ensured as well that equations (18) to (21) retain their validity, with ϕ in (18)–(20) replaced by g as in (23).

EQUATIONS FOR BEAMS WITH ONE CROSS SECTIONAL AXIS OF SYMMETRY

For cases for which there is geometrical and material symmetry about one of the two cartesian cross sectional axes, say the axis of y, we will have $g(x, y) = -g(-x, y)$ and in (21)

$$S_{xG} = 0, \quad S_{xE} = 0, \quad K_E = 0, \quad \Gamma_{yE} = 0, \tag{25}$$

and eqns (8)–(10) decouple into one elementary system for F, Q, N, w, v, β, and one system which is of interest within the present context, with equilibrium equations

$$M' - P + m = 0, \quad P' + p = 0, \quad T' + t = 0, \quad R' - S + r = 0. \tag{26}$$

As far as the constitutive equations for the system (26) are concerned we have, when $g = \phi$, on the basis of (21)

$$\begin{bmatrix} M \\ R \end{bmatrix} = \begin{bmatrix} I_{xE} & \Gamma_{xE} \\ \Gamma_{xE} & \Gamma_{\phi E} \end{bmatrix} \begin{bmatrix} \alpha' \\ \lambda' \end{bmatrix}, \quad \begin{bmatrix} P \\ T \\ S \end{bmatrix} = \begin{bmatrix} A_G & -S_{yG} & S_{yG} \\ -S_{yG} & I_G & -J_G \\ S_{yG} & -J_G & J_G \end{bmatrix} \begin{bmatrix} u'+\alpha \\ \theta' \\ \lambda \end{bmatrix}. \tag{27}$$

The choice of g corresponding to (23) is for the case of symmetry about the y-axis†

$$g = c_1 x + c_5 xy, \tag{28}$$

with the equations determining c_1 and c_5 being, in accordance with (24)

$$c_1 A_G + c_5 S_{yG} = S_{yG}, \quad c_1 S_{yG} + c_5 I_G = I_{yG} - I_{xG}, \tag{29}$$

and therewith

$$c_1 = \frac{2 S_{yG} I_{xG}}{A_G I_G - S_{yG}}, \quad c_5 = \frac{A_G(I_{yG} - I_{xG}) - S_{yG}^2}{A_G I_G - S_{yG}^2}. \tag{30}$$

Inasmuch as $\int G g_{,x}\, dA = S_{yG}$ not only when $g = \phi$ but, in view of the form of (28) and (29), also when g is given by (28), and in view of the fact that for both choices of g we have that $C_{\lambda\lambda} = -C_{\theta\lambda}$ we may, with $C_{\theta\lambda}$ in accordance with (11) and (12), use (27) also for the case that g is as in (28) and (29) upon writing

$$J_G = c_5(I_{yG} - I_{xG}) + c_1 S_{yG}, \tag{31}$$

and upon writing

$$\Gamma_{xE} = c_5 \int E x^2 y \, dA + c_1 I_{xE}, \tag{32a}$$

$$\Gamma_{\phi E} = c_5^2 \int E x^2 y^2 \, dA + 2 c_1 c_5 \int E x^2 y \, dA + c_1^2 I_{xE}. \tag{32b}$$

We note that for the case of a *doubly symmetric cross section* we have $S_{yG} = 0$, $\int E x^2 y \, dA = 0$ and $c_1 = 0$, and therewith

$$J_G = \frac{(I_{yG} - I_{xG})^2}{I_G}, \quad \Gamma_{xE} = 0, \quad \Gamma_{\phi E} = \left(\frac{I_{yG} - I_{xG}}{I_G}\right)^2 \int E x^2 y^2 \, dA. \tag{33}$$

It is apparent, in view of the assumed form of g, that for a beam with *uniform elliptical cross* section the value of J_G in (33) will coincide with the value of J_G in (18).

SEQUENTIAL DETERMINATION OF STATICAL AND KINEMATICAL QUANTITIES

For cases for which the external load terms in the equilibrium equations (8) and (9) are given functions of z it is useful to have constitutive equations for displacements and displacement derivatives in terms of statical quantities by inversion of (10) and (11), in the form

$$\begin{bmatrix} w' \\ \alpha' \\ \beta' \\ \lambda' \end{bmatrix} = \begin{bmatrix} D^{-1}_{FF} & D^{-1}_{FM} & D^{-1}_{FN} & D^{-1}_{FR} \\ D^{-1}_{MF} & D^{-1}_{MM} & D^{-1}_{MN} & D^{-1}_{MR} \\ D^{-1}_{NF} & \cdot & \cdot & \cdot \\ D^{-1}_{RF} & \cdot & \cdot & D^{-1}_{RR} \end{bmatrix} \begin{bmatrix} F \\ M \\ N \\ R \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} u'+\alpha \\ v'+\beta \\ \theta' \\ \lambda \end{bmatrix} = \begin{bmatrix} C^{-1}_{PP} & \cdot & \cdot & C^{-1}_{PS} \\ C^{-1}_{QP} & \cdot & \cdot & C^{-1}_{QS} \\ C^{-1}_{TP} & \cdot & \cdot & C^{-1}_{TS} \\ C^{-1}_{ST} & \cdot & \cdot & C^{-1}_{SS} \end{bmatrix} \begin{bmatrix} P \\ Q \\ T \\ S \end{bmatrix} \tag{35}$$

†This choice of g corresponds effectively to what has been done in the analysis of plate-like beams in [3].

With (34) and (35) we may complement the seven equilibrium equations for eight statical quantities by an eighth equation containing statical quantities only, by using the expressions for λ' and λ in (34) and (35) for the purpose of deducing a *compatibility* relation

$$D_{RF}^{-1}F + D_{RM}^{-1}M + D_{RN}^{-1}N + D_{RR}^{-1}R = (C_{SP}^{-1}P + C_{SQ}^{-1}Q + C_{ST}^{-1}T + C_{SS}^{-1}S)', \tag{36}$$

with this relation becoming a differential equation for R upon eliminating S through the use of (9d), of the form

$$(C_{SS}^{-1}R')' - D_{RR}^{-1}R = D_{RF}^{-1}F + D_{RM}^{-1}M + D_{RN}^{-1}N - (C_{SP}^{-1}P + C_{SQ}^{-1}Q + C_{ST}^{-1}T + C_{SS}^{-1}r)'. \tag{37}$$

Having then F, P, Q, T, M, N and R from (8a–c), (9a–c) and (37) we may obtain w, α, β, θ, u, v by direct integration of the relevant equations in (34) and (35), with $S = R' + r$ on the right hand side of (35) and with λ given by the last equation in (35) without integration.

ASYMPTOTIC DETERMINATION OF INTERIOR SOLUTION

By interior solution of the given problem we here mean a solution which coincides with the actual solution outside of (narrow) edge zones associated with the enforcement of warping constraint boundary conditions. Mathematically, the possibility of a distinction between edge zone and interior domain is given, on the basis of equation (37), upon stipulating an order of magnitude relation

$$D_{RR} \ll C_{SS}L^2, \tag{38}$$

in which L is a representative axial length, which may be the span of the beam, or the distance over which significant changes of loads are occurring.

For cases for which (38) applies it is of advantage to write the solution (37) in the form

$$R = R_e + R_i, \tag{39}$$

with R_e being the general solution of the homogeneous equation

$$(C_{SS}^{-1}R_e)'' - D_{RR}^{-1}R_e = 0, \tag{40}$$

and with R_i approximated by the expression

$$D_{RR}^{-1}R_i = (C_{SP}^{-1}P + C_{SQ}^{-1}Q + C_{ST}^{-1}T + C_{SS}^{-1}r)' - D_{RF}^{-1}F - D_{RM}^{-1}M - D_{RN}^{-1}N. \tag{41}$$

Having equations (40) and (41) we may then obtain expressions for w, α and β from (34), in the form

$$(w, \alpha, \beta) = (w_i, \alpha_i, \beta_i) + \int (D_{FR}^{-1}, D_{MR}^{-1}, D_{NR}^{-1})R_e \, dz, \tag{42}$$

with the meaning of w_i, α_i, β_i being evident from (34), (39) and (41). Alongside (42) we have, from (35) in conjunction with (9-d) and (39)

$$\lambda = \lambda_i + C_{SS}^{-1}R_e', \tag{43}$$

where it remains to utilize (42) and (43) together with the order of magnitude relation

$$R_e = o(LR_e'), \tag{44}$$

for the purpose of deducing from a set of four boundary conditions for w, α, β, λ a *contracted* set of three conditions for the determination of w_i, α_i, β_i, λ_i.

To obtain this set of contracted boundary conditions we transform the integral involving R_e in (42) with the help of the differential equation (40) as follows

$$\int D^{-1}R_e\,dz = \int D^{-1}D_{RR}(C_{SS}^{-1}R_e)''\,dz = D^{-1}D_{RR}(C_{SS}^{-1}R_e)' - \int (D^{-1}D_{RR})'(C_{SS}^{-1}R_e)'\,ds. \quad (45)$$

It may be concluded from (45) and (44) that

$$\int D^{-1}R_e\,dz \approx D^{-1}D_{RR}C_{SS}^{-1}R_e', \quad (46)$$

except for quantities which are small of higher order. This being the case we may further conclude from (42) and (43), by eliminating the remaining *dominant* terms with R_e' that, in generalization of a special form of this result in [3], the four boundary conditions

$$w = \bar{w}, \quad \alpha = \bar{\alpha}, \quad \beta = \bar{\beta}, \quad \lambda = \bar{\lambda}, \quad (47)$$

are in fact equivalent to three *contracted* conditions for the interior solution contribution, of the form

$$(D_{FR}w_i, D_{MR}\alpha_i, D_{NR}\beta_i) - D_{RR}\lambda_i = (D_{FR}\bar{w}, D_{MR}\bar{\alpha}, D_{NR}\bar{\beta}) - D_{RR}\bar{\lambda}. \quad (48)$$

Proceeding in an analogous way in regard to the conditions $u = \bar{u}$, $v = \bar{v}$, $\theta = \bar{\theta}$ we find that here the order of magnitude relation (44) implies that the conditions for u, v, θ are asymptotically equivalent to the *reduced* conditions

$$u_i = \bar{u}, \quad v_i = \bar{v}, \quad \theta_i = \bar{\theta}, \quad (49)$$

for the interior solution contribution.

ASYMPTOTIC DETERMINATION OF SHEAR CENTER LOCATION

By asymptotic determination we mean here a determination of the coordinates of the center of shear based on the interior portion of the solution of the problems of torsion and flexure in the sense of the discussion in [3].

We begin with the relation

$$\theta' = C_{TP}^{-1}P + C_{TQ}^{-1}Q + C_{TT}^{-1}T + C_{TS}^{-1}S, \quad (35')$$

and in this write $S = R'$. We then take $R' = R_i' + R_e' \approx R_i'$ with $r = 0$, from (41) where we restrict ourselves for present purposes to the case of axially homogeneous beams, so as to have from (34), with (8b, c),

$$S = -(D_{RR}/D_{RM})P - (D_{RR}/D_{RN})Q, \quad (50)$$

and then from (35')

$$\theta' = \left(\frac{1}{C_{TP}} - \frac{D_{RR}}{C_{TS}D_{RM}}\right)P + \left(\frac{1}{C_{TQ}} - \frac{D_{RR}}{C_{TS}D_{RN}}\right)Q + \frac{T}{C_{TT}}. \quad (51)$$

Equation (51) simplifies considerably upon consideration of the fact that with the C-matrix as in (21) we have from (11)

$$T + S = (I_G - J_G)\theta', \quad (52)$$

and therewith in accordance with (35),

$$\frac{1}{C_{TP}} = \frac{1}{C_{TQ}} = 0, \quad \frac{1}{C_{TT}} = \frac{1}{C_{TS}} = \frac{1}{I_G - J_G}, \quad (53)$$

and then from (51)

$$\theta' = \frac{1}{C_{TT}}\left(T - \frac{D_{RR}}{D_{RM}} P - \frac{D_{RR}}{D_{RN}} Q\right). \tag{54}$$

Having (54) we define shear center coordinates x_s, y_s consistent with the procedure in [1] and [4] by the stipulation

$$\theta' = 0, \quad T = Qx_s - Py_s, \tag{55}$$

and this, in conjunction with (54), results in the formulas

$$x_s = \frac{D_{RR}}{D_{RN}}, \quad y_s = -\frac{D_{RR}}{D_{RM}}. \tag{56}$$

A determination of D_{RM}, D_{RN} and D_{RR} in accordance with equations (34) and (10), with the four by four matrix in (10) as in (21), then gives as expressions for x_s and y_s,

$$\begin{aligned} x_s &= \Delta^{-1}[-\Gamma_{yE}(I_{xE} - A_E^{-1}S_{xE}^2) + \Gamma_{xE}(K_E - A_E^{-1}S_{xE}S_{yE})], \\ y_s &= \Delta^{-1}[\Gamma_{xE}(I_{yE} - A_E^{-1}S_{yE}^2) - \Gamma_{yE}(K_E - A_E^{-1}S_{xE}S_{yE})], \end{aligned} \tag{57}$$

with the denominator Δ being of the form

$$\Delta = (I_{xE} - A_E^{-1}S_{xE}^2)(I_{yE} - A_E^{-1}S_{yE}^2) - (K_E - A_E^{-1}S_{xE}S_{yE})^2. \tag{58}$$

For the case $S_{xE} = S_{yE} = 0$ these formulas reduce to the corresponding formulas in [1]. Remarkably, the values of x_s and y_s in (57) show no explicit dependence on variations of G over the cross section. There is, however, an implicit dependence, inasmuch as the integrands in the defining relations for Γ_{xE} and Γ_{yE} depend on the distribution of G, in accordance with the form of the differential equation (13) for ϕ, or in accordance with the defining relations (23), (24) and (17') for the approximating function g.

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